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A NOTE ON UNIFORM DISTRIBUTION FOR PRIMES AND CLOSED ORBITS

BY

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ABSTRACT

In this note we study an analogue of Vinogradov's uniform distribution result for prime numbers in the context of hyperbolic flows and their closed orbits. We obtain estimates for the Hausdorff dimension of certain exceptional sets.

§0. Introduction

For a sequence of real numbers in the unit interval there exists a natural idea of the sequence being uniformly distributed. If $0 < \alpha < 1$ is irrational then the sequence $\alpha n \pmod 1$ is easily shown to be uniformly distributed, where n runs through the natural numbers [14]. In 1948 Vinogradov proved the harder result that $\alpha p \pmod 1$ is uniformly distributed where p runs through the prime numbers [24]. This gives information on the distribution of the primes (Dirichlet's theorem can be interpreted as a result on the distribution of $(r/s)p \pmod 1$ over the rationals in the unit interval with denominator s, when r/s is a rational).

The main purpose of this paper is to study the length spectrum of closed orbits for certain dynamical systems, motivated by Vinogradov's result and the close prime-closed orbit correspondence exhibited in [15].

In section 1 we present Vinogradov's theorem. The proof given is not essentially new (cf. [23], [7]) but is included for completeness, in the absence of an appropriate reference.

In section 2 we consider the size of the exceptional set of α for which the Weyl sums don't converge at certain rates. This material is independent of later sections, although it serves to motivate the formulation of subsequent results, and is included for its independent interest.

In the final section we come to the crux of the paper. In this section we replace

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the primes by the norms of closed orbits for Axiom A flows (this is precisely the correspondence between primes and closed orbits occurring in prime orbit theorems [15]). We then consider the size of the exceptional set of $0<\alpha<1$ for which $\alpha\langle\text{Norms}\rangle \pmod 1$ is not uniformly distributed. We also take the opportunity to present additional results for closed geodesics on compact surfaces of curvature -1.

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§1. Uniform distribution and prime numbers

Let $(x_n)_{n=1}^{\infty}$ be a sequence in the interval $[0,1]$ and let μ_N be the (purely atomic) probability measure constructed by equidistributing measure over the first N terms in the sequence, i.e. $\mu_N=(1/N)\Sigma_{n=1}^{N}\delta_{x_n}$ where δ_{x_n} is the probability measure consisting of a single atom at x_n. The sequence $(x_n)_{n=1}$ is then called *uniformly distributed* if μ_N converges to Lebesgue measure in the weak* topology. In particular, this is equivalent to $\int_0^1 e(kx)d\mu_N(x)\to 0$ as $N\to+\infty$, for each $k\in Z\backslash\{0\}$, where we have used the notation $e(z)=\exp 2\pi iz$. (This is exactly Weyl's criterion [14].)

We use $f(t)\ll g(t)$ (or sometimes $f(t)=O(g(t))$) to denote that $f(t)/g(t)$ is bounded for large $t>0$).

THEOREM 1. (Vinogradov) *Let $(p_n)_{n=1}^{\infty}$ be the sequence of prime numbers and let $0<\alpha<1$ be irrational, then $(\alpha p_n \pmod 1)_{n=1}^{\infty}$ is uniformly distributed.*

PROOF. Given $\tau>0$, choose a,q to be coprime positive integers (i.e. $(a,p)=1$) such that $|\alpha-a/q|<1/\tau q$ and $q\leqq\tau$. We can write $\alpha=a/q+\theta/\tau q$ with $|\theta|<1$ (cf. [18]).

We require estimates on the partial sums $s(N)=\Sigma_{p_n\leq N}e(\alpha p_n)$ for $N>0$, so in particular we will take $\tau=N\exp-(\log N)^{1/6}$.

Case I. $1\leqq q\leqq(\log N)^{20}$.

Divide $[1,N]$ into intervals of length $A=N/\exp(\log N)^{1/4}$. For a prime $1\leqq p\leqq N$ assume p lies in one such interval $[N_1-A,N_1]$. Then

$$\begin{aligned}|p(\alpha-a/q)-\theta N_1/q\tau| &\ll|\theta|\cdot|p-N_1|/q\tau\\ &\leqq 1\cdot A/1\cdot\tau\\ &\ll A\exp(\log N)^{1/6}/N\end{aligned}$$

and the final term tends to zero as N increases. Therefore

$$\left|\sum_{N_1-A<p_n\leqq N_2} e(\alpha p_n)\right| \ll \left|\sum_{N_1-A<p_n\leqq N_1} e\left(\frac{a}{q}\cdot p_n\right)\right| + A\cdot[A\exp(\log N)^{1/6}]/N$$

since the summation is over less than A terms and $\theta N_1/q\tau$ is a common argument to these terms.

Estimate 1. Let $1\leqq l\leqq q$, $(l,q)=1$, $q\leqq(\log N)^{20}$ then there exists $C>0$ such that

$$\pi(N,l,q)=\pi(N)/\phi(q)+O(N\exp-C(\log N)^{1/2})$$

where $\pi(N)=\operatorname{Card}\{p_n \mid p_n\leqq N\}$,

$$\pi(N,l,q)=\operatorname{Card}\{p_n\leqq N \mid p=l \ (\mathrm{mod}\ q)\},$$
$$\phi(q)=\operatorname{Card}\{1\leqq l\leqq q \mid (l,q)=1\}$$

(cf. [6], p. 133).

We use the above estimate as follows: if

$$s_1=\sum_{N_1-A<p_n\leqq N_1} e\left(\frac{a}{q}p_n\right)$$

then

$$|s_1|\ll\left|\sum_{\substack{1\leqq l\leqq q\\(l,q)=1}} e\left(\frac{a}{q}\cdot l\right)[\pi(N_1,l,q)-\pi(N_1-A,l,q)]\right|$$

$$\ll\left|\sum_{\substack{1\leqq l\leqq q\\(l,q)=1}} e\left(\frac{a}{q}\cdot l\right)\right|\left[\frac{\pi(N_1)-\pi(N_1-A)}{\phi(q)}+N\exp-C(\log N_1)^{1/2}\right] \tag{1.1}$$

$$\ll\left|\sum_{\substack{1\leqq l\leqq q\\(l,q)=1}} e\left(\frac{a}{q}\cdot l\right)\right|\left[\frac{\pi(N_1)-\pi(N_1-A)}{\phi(q)}\right]+qN\exp-C(\log N_1)^{1/2}.$$

Estimate 2. For $\varepsilon>0$, $a>0$ we have

$$\left|\sum_{\substack{1\leqq l\leqq q\\(l,q)=1}} e\left(\frac{a}{q}\cdot l\right)\right|\ll q^{\varepsilon} \qquad \text{(cf. [11], p. 1).}$$

Applying Estimate 2 to (1.1) and recalling the hypothesis $q\leqq(\log N)^{20}$ gives

$$|s_1|\ll q^{\varepsilon}\left(\frac{\pi(N_1)-\pi(N_1-A)}{\phi(q)}\right)+N(\log N)^{20}\exp-C(\log N_1)^{1/2}$$

$$\ll q^{\varepsilon}\left(\frac{\pi(N_1)-\pi(N_1-A)}{\phi(q)}\right)+N(\log N)^{20}\exp-C[\log N-(\log N)^{1/4}]^{1/2}$$

(since $\log N_1\gg\log A\gg\log N-(\log N)^{1/4}$).

We can now sum over the N/A partial sums of the form s_1 to get

$$S(N) \ll q^{\varepsilon}\pi(N)/\phi(q) + (\log N)^{20}\frac{N^2}{A}\exp - C[\log N - (\log N)^{1/4}]^{1/2}$$

and so

$$S(N)/\pi(N) \ll q^{\varepsilon}/\phi(q) + (\log N)^{20+1}\cdot \exp((\log N)^{1/4} - C[\log N - (\log N)^{1/4}]^{1/2}).$$

Estimate 3. For $\varepsilon > 0$, $q^{\varepsilon}/\phi(q) \to 0$ as $q \to +\infty$ (cf. [10], §18.4).

Therefore we have $S(N)/\pi(N) \to 0$ as $N \to +\infty$ since q must take larger and larger values.

Case II. $(\log N)^{20} \leqq q \leqq \tau = N \exp - (\log N)^{1/6}$.

We can derive the necessary bounds from the following estimate.

Estimate 4. Assume $|\alpha - a/q| < 1/q^2$ then

$$S(N) = \sum_{p_n \leqq N} e(\alpha p_n) \ll (Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2})(\log N)^4$$

(cf. [6], p. 143).

The above estimate, together with the hypothesis on q, immediately gives

$$\frac{S(N)}{N/\log N} \ll (q^{-1/2} + N^{-1/5} + N^{-1/2}q^{1/2})(\log N)^5$$

$$\to 0 \qquad \text{as } N \to +\infty.$$

Of course, since $N/\log N = O(\pi(N))$ by the prime number theorem we have that in either of the two cases $S(N)/\pi(N) \to 0$ as $N \to +\infty$ and thus by Weyl's criterion $(\alpha p_n \pmod 1)_{n=1}^{\infty}$ is uniformly distributed.

§2. Rates of convergence

From the proof of Theorem 1 it is simple to see that the Weyl sum $S(N)/\pi(N)$ satisfies $S(N)/\pi(N) = O((\log N)^{-k})$, for any $k > 0$. (To see this replace $(\log N)^{20}$ by $(\log N)^{2k}$ in the bounds on the two cases, $k \geqq 10$.) However, the second term on the right-hand side of Estimate 4 means that the fastest rate of convergence, for any choice of α, which can be expected from this proof is $S(N)/\pi(N) \ll (\log N)^5 N^{-1/5}$. For a given $0 < \delta \leqq 1/5$ and irrational α, Estimate 4 gives that $S(N)/\pi(N) \ll (\log N)^5 N^{-\delta}$ provided the following condition is satisfied: *For all sufficiently large N there exists $N^{2\delta} < q < N^{1-2\delta}$ and $1 < p < q$, $(p,q) = 1$ with $|\alpha - p/q| < 1/q^2$.* Using work of W. Schmidt [19] it can be shown that for any given $0 < \delta \leqq 1/5$ the above condition is satisfied for almost all $0 < \alpha < 1$ (with respect to Lebesgue measure).

In order to describe the "size" of sets of zero Lebesgue measure it is appropriate to introduce the notion of Hausdorff dimension. Let $\delta, \rho > 0$ and let $S \subseteq R$ and define $m_\delta^\rho(S) = \inf_{\mathscr{C}} \Sigma_{C \in \mathscr{C}} (\operatorname{diam} C)^\rho$, where $\mathscr{C}$ runs over all countable covers of S by closed sets of diameter less than δ. Let $H^\rho(S) = \lim_{\delta \to 0} m_\delta^\rho(S)$, then there exists $1 \geqq \beta \geqq 0$ such that $H^\rho(S)$ is infinite for $\rho > \beta$ and zero for $\rho < \beta$. We call β the *Hausdorff dimension* of δ and denote it by $\mathrm{HD}(S)$ (cf. [3]). A set of fractional Hausdorff dimension has zero Lebesgue measure and a countable union of points has zero Hausdorff dimension.

PROPOSITION 1. *Let $0 < \gamma < \beta$ and let*

$$E = \{\alpha \mid \exists \text{ arbitrarily large } N \text{ such that } \forall N^\gamma \leqq q \leqq N^\beta, \\ (p,q) = 1, p < q, |\alpha - p/q| > 1/q^2\}$$

then $\mathrm{HD}(E) \leqq 2\gamma/(\gamma + \beta)$.

PROOF. Let

$$E_N = \bigcup_{q = N^\gamma}^{N^\beta} \bigcup_{\substack{p=1 \\ (p,q)=1}}^{q} (p/q - 1/q^2, p/q + 1/q^2),$$

then we can write

$$E = \bigcap_{M=1}^{\infty} \bigcup_{N=M}^{\infty} E_N^c.$$

It is convenient to introduce *Ford circles* (Fig. 1). These are circles in the upper half plane tangent to the real line at p/q, $(p,q) = 1$, $q \geqq 1$ and of diameter $1/q^2$ (cf. [18], [22]).

Furthermore, the interiors of two such circles never intersect and circles corresponding to p/q and p_0/q_0 touch if and only if $|pq_0 - p_0q| = 1$.

The set E_N is the projection onto R of those circles corresponding to $N^\gamma \leqq q \leqq N^\beta$.

Step 1. Assume $q_1 < N^\beta$, where $N > 0$. By considering the relative sizes of any three mutually tangent Ford circles we can choose $C > 0$ (independent of q_1, γ, N) such that the nearest q-circle to the base of a q_1-circle, with $N^\gamma \leqq q \leqq N^\beta$, is based at a distance $1/qq_1 \leqq C/q_1N^\beta$ (Fig. 2).

In particular, the intervals about p_1/q_1 in E_N are covered by intervals $(p_1/q_1 - C/q_1N^\beta,\ p_1/q_1 + C/q_1N^\beta)$.

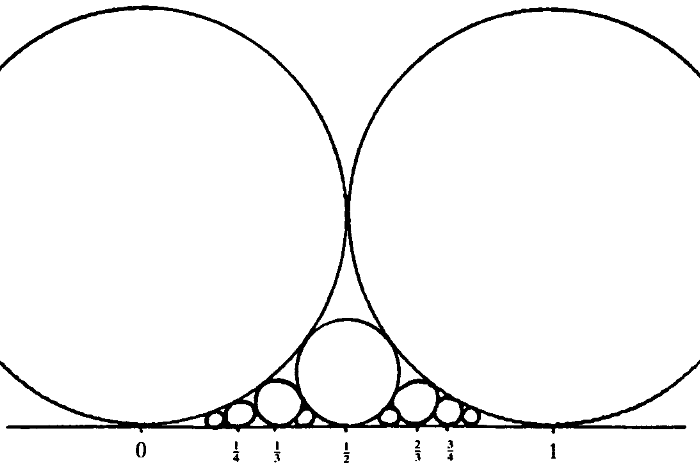

Fig. 1.

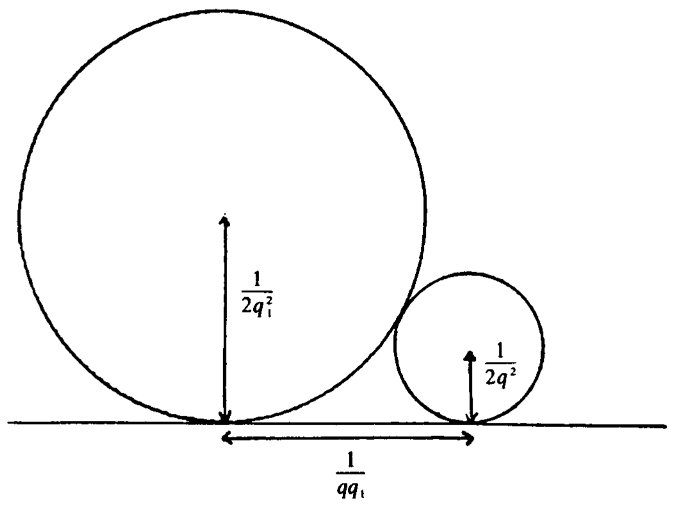

Fig. 2.

Step 2. We shall now show that intervals of the above form cover all of E_N. Take $q_1 = [N^\gamma] - 1$ and consider (one side of) a particular q_1-circle. Choose the smallest q-circle touching both R and this q_1-circle subject to the condition $N^\gamma \leqq q \leqq N^\beta$. Next inductively choose successive q'-circles of increasing diameter with $q_1 \leqq q' \leqq N^\beta$ such that each new circle touches the previous circle and the original q_1-circle. Continue until it becomes impossible to satisfy $q' \geqq q_1$ (Fig. 3).

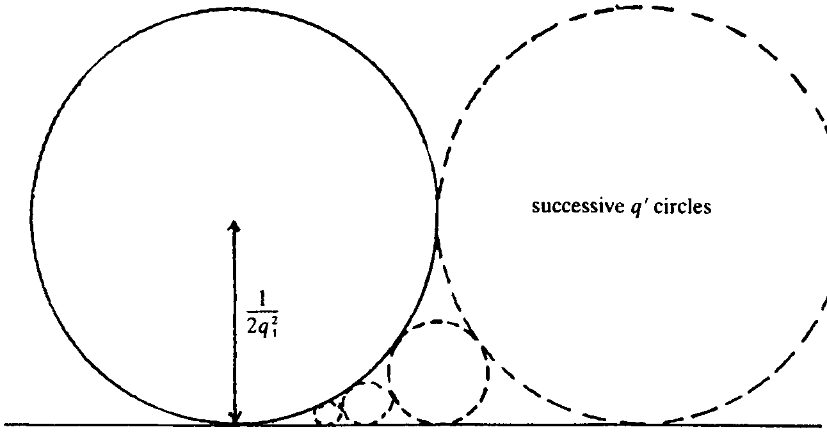

Fig. 3.

It is obvious that $E_N \cap (p_1/q_1 - 1/q_1^2, p_1/q_1 + 1/q_1^2)$ is covered by $(p_1/q_1 - C/q_2N^\beta, p_1/q_1 + C/q_1N^\beta)$.

Step 3. We can now repeat the above argument in step 2 with $q_1 = [N^\gamma]-2, [N^\gamma]-3, \ldots, 2$ to deduce that E_N is covered by intervals of the form $(p_1/q_1 - C/q_1N, p_1/q_1 + C/q_1N)$ where $p_1 \leqq q_1$, $(p_1, q_1) = 1$.

Step 4. Given $\varepsilon > 0$, choose M sufficiently large that the intervals $(p_1/q_1 - C/q_1N^\beta, p_1/q_1 + C/q_1N^\beta)$ satisfying $q_1 \leqq N^\gamma$, $(p_1, q_1) = 1$, $p_1 < q_1$ form an ε-covering for E_N (for any $N \geqq M$). Denote the covering for E_N by $\mathscr{C}_N$, then for $1 \geqq \rho \geqq 0$:

$$\sum_{\hat{C} \in \mathscr{C}_N} (\operatorname{diam} \hat{C})^\rho = \sum_{q_1 \leqq N^\gamma} \sum_{\substack{(p_1, q_1)=1 \\ p_1 \leqq q_1}} \left(\frac{2C}{q_1 N^\beta}\right)^\rho$$

$$\leqq \left(\frac{2C}{N^\beta}\right)^\rho \sum_{q_1 \leqq N^\gamma} q_1^{1-\rho} \tag{2.1}$$

$$\leqq \left(\frac{2C}{N^\beta}\right)^\rho N^\gamma \cdot N^{(1-\rho)\gamma}.$$

If $\rho > (2\gamma + 1)/(\gamma + \beta)$ then the exponent of N in (2.1) is less than -1. If we write $E = \bigcap_{M=1}^\infty \bigcup_{N=M}^\infty E_N$ then $\mathscr{C} = \bigcup_{N>M} \mathscr{C}_N$ is obviously an ε-cover and $\Sigma_{C \in \mathscr{C}} (\operatorname{diam} C)^\rho \leqq \Sigma_{N \geqq M} 1/N^\varepsilon < +\infty$, for $\varepsilon = \rho(\gamma + \beta) - 2\gamma > 1$ (by comparison with the Riemann zeta function).

This shows that $\mathrm{HD}(E) \leqq (2\gamma + 1)/(\gamma + \beta)$. However, we can scale γ and β by the same factor without changing the Hausdorff dimension of E. This proves the proposition.

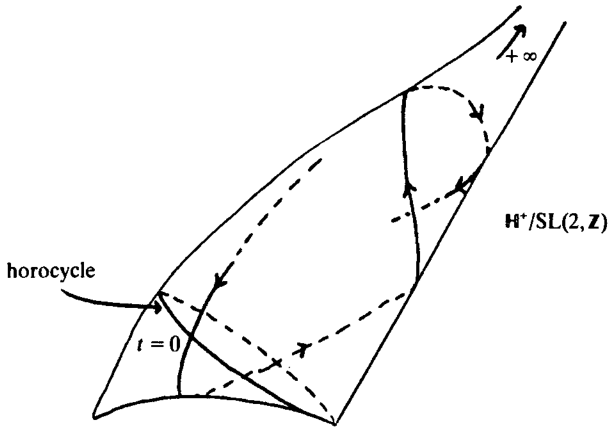

Fig. 4.

By combining the above proposition with earlier remarks we get the following

PROPOSITION 2. *For any* $k>0$, $S(N)/\pi(N) \ll (\log N)^{-k}$. *Furthermore, the union of all irrational* $0<\alpha<1$ *which do* not *satisfy* $S(N)/\pi(N) \ll N^{-\delta}$ *(for given* $0<\delta\leqq 1/5$*) has Hausdorff dimension at most* $1-4\delta$.

REMARK. Proposition 1 can be interpreted geometrically for the modular surface as follows: Given $\eta>0$ let G_η be the union of all geodesics coming from infinity which, after first crossing the horocycle through the ramification point of order 2 at time $t=0$, stay on the cusp side of the horocycle for stretches $t_i \leqq t \leqq (1+\eta)t_i$, where $t_i>0$ are arbitrarily large. Then the Hausdorff dimension of G_η is bounded above by $1+1/(1+\eta/2)$ (cf. [22]).

REMARK. It has been pointed out to the author that an alternative derivation of Proposition 1 is possible based on the work of Besicovitch (J. London Math. Soc. **9** (1934), 126–131).

§3. Axiom A and geodesic flows

Let ϕ be a C^1-flow on a compact manifold M. A compact invariant set Λ containing no fixed points is called *hyperbolic* if the unit tangent bundle over Λ splits into the Whitney sum of three $D\phi$-invariant continuous sub-bundles $T_\Lambda M = E + E^s + E^u$, where E is the one-dimensional bundle tangent to the flow, and there are constants $C, \lambda>0$ such that

(a) $\|D\phi_t(v)\| \leqq Ce^{-\lambda t}\|v\|$ for $v\in E^s$, $t\geqq 0$,

(b) $\|D\phi_{-t}(v)\| \leqq Ce^{-\lambda t}\|v\|$ for $v\in E^u$, $t\geqq 0$.

A hyperbolic set Λ is called *basic* if

(a) the periodic orbits of $\phi_t \mid$ are dense in Λ,

(b) $\phi_t|_\Lambda$ is topologically transitive,

(c) there is an open set $U \supset \Lambda$ with $\Lambda = \bigcap_{t=-\infty}^{+\infty} \phi_t U$.

We shall be interested in the flow $\phi_t : \Lambda \to \Lambda$ which is essentially the case of an *Axiom A flow*. The flow ϕ is called (*topologically*) *weakmixing* if there is no non-trivial solution to $F\phi_t = e^{iat}F$, where $F \in C(\Lambda)$ and $a > 0$.

A special case of an Axiom A flow is a geodesic flow on (the unit tangent bundle T_1M of) a compact surface M of constant negative curvature, $\kappa = -1$, say. (Here M is topologically a g-holed torus, $g \geqq 2$.)

The flow $\phi_t : T_1M \to T_1M$ is defined by choosing for each $(x, v) \in T_1M$ the unique geodesic $\gamma : R \to M$ with $(\gamma(0), \dot\gamma(0)) = (x, v)$ and taking $\phi_t(x, v) = (\gamma(t), \dot\gamma(t)) \in T_1M$. That the geodesic flow is a weak-mixing Axiom A flow (in fact Anosov) of entropy $h(\phi) = 1$ is shown in [1].

For a geodesic flow ϕ a closed ϕ-orbit in T_1M projects down to a closed geodesic γ in M of length $l(\gamma)$. Furthermore, there is a one-one correspondence between closed geodesics and free homotopy classes for M.

We want the word length of a closed geodesic to reflect the number of generators of the fundamental group in the corresponding free homotopy class. To avoid ambiguities we choose to fix a choice of generators for the covering group. This gives a canonical copy of the surface in the Universal cover called the fundamental domain R (bounded by the isometric circles of the generators cf. [13]). A closed geodesic is said to have word length $w(\tau)$ if its lift to the cover intersects $w(\tau)$ copies of R (generated by the covering group).

A canonical example of an Axiom A flow can be constructed as follows. Let A be an irreducible $k \times k$ matrix with zero-one entries and define

$$\Sigma_A = \left\{ x \in \prod_{-\infty}^{+\infty} \{1, 2, \dots, k\} \,\middle|\, A(x_n, x_{n+1}) = 1 \right\}.$$

We define a metric on Σ_A by

$$d(x, y) = \sum_{n=-\infty}^{+\infty} \frac{e(x_n, y_n)}{2^{|n|}} \qquad \text{where } e(x_n, y_n) = \begin{cases} 1 & \text{if } x_n = y_n, \\ 0 & \text{otherwise.} \end{cases}$$

With this metric Σ_A is a compact zero-dimensional space and define a homeomorphism $\sigma : \Sigma_A \to \Sigma_A$, called a *subshift of finite type*, by $(\sigma x)_n = x_{n+1}$. Let $f : \Sigma_A \to R^+$ be a strictly positive Hölder continuous function and define

$$\Sigma_A^f = \{(x, t) \in \Sigma_A \times R \mid 0 \leqq t \leqq f(x)\}$$

where $(x,f(x))$ and $(\sigma x,0)$ are identified. The *suspended flow* $\sigma_t^f:\Sigma_A^f\to\Sigma_A^f$ is defined locally by $\sigma_t^f(x,r)=(x,r+t)$.

PROPOSITION 3. (Bowen) (i) *Any Axiom A flow restricted to a one-dimensional basic set is conjugate to a suspended flow, and vice versa* [4].

(ii) *For any Axiom A flow restricted to a basic set* $\phi:\Lambda\to\Lambda$ *there exists a suspended flow* $\sigma_t^f:\Sigma_A^f\to\Sigma_A^f$ *and a bounded-one continuous surjection* $\pi:\Sigma_A^f\to\Lambda$ *such that* $\phi_t\pi=\pi\sigma_t^f$ [5].

For a geodesic flow it is possible to choose the suspended flow σ^f so that a closed σ-orbit $\{x,\sigma x,\ldots,\sigma^{n-1}x\}$ corresponds to a closed geodesic γ with both $l(\gamma)=f^n(x)$ and $\omega(\gamma)=n$ [20], [16].

For an Axiom A flow $\phi:\Lambda\to\Lambda$ of topological entropy $h=h(\phi)$ let τ denote a closed ϕ-orbit and let $\lambda(\tau)$ be its least period. For $t>0$, let $\pi(t)=\mathrm{Card}\{\tau\mid\exp h\lambda(\tau)\leqq t\}$.

We write $f(t)\sim g(t)$ if $f(t)/g(t)\to 1$ as $t\to+\infty$. The following was proved in [15].

PROPOSITION 4 (Prime orbit theorem). *For any weak-mixing Axiom A flow (restricted to a basic set)* $\pi(t)\sim t/\log t$.

The above proposition is analogous to the prime number theorem (even up to the use of a zeta function in its proof).

REMARK. A version of Proposition 4 was originally proved by Huber for geodesic flows on compact surfaces with $\kappa=-1$ [13]. In fact, in this special case he showed that there exists $0<\delta<1$ with $\pi(t)=\mathrm{li}(t)+O(t^\delta)$. (The value of δ is intimately connected with the geometry of the surface M.)

§4. Uniform distribution and closed orbits

Continuing the analogy between primes and closed orbits alluded to in section 3 we consider those α for which $(\alpha\exp h\lambda(\tau)\pmod 1)$ is not uniformly distributed.

Let $0<\delta<t$ and define

$$\Phi^\delta(t)=\sup\{\mathrm{Card}[\tau\mid r\leqq\exp h\lambda(\tau)\leqq r+\delta]\mid 0\leqq r\leqq t-\delta\}.$$

That is, $\Phi^\delta(t)$ is the maximum amount of clustering of $\exp h\lambda(\tau)\leqq t$ in intervals of length δ. We consider below the extent to which this clustering effects the size of the exceptional set $E=\{\alpha\mid(\alpha\exp h\lambda(\tau)\pmod 1)$ is *not* uniformly distributed$\}$.

PROPOSITION 5. (a) *If there exists $\varepsilon > 0$ such that for some $\delta > 0$, $\Phi^\delta(t) \ll t/(\log t)^{1+\varepsilon}$ then E has zero Lebesgue measure.*

(b) *If there exists $0 < \theta < 1$ such that for some $\delta > 0$, $\Phi^\delta(t) \ll t^\theta$, then the Hausdorff dimension of E is bounded above by θ.*

PROOF. (a) This is a classical result due to Weyl [25] (cf. [2]).

(b) Our proof borrows ideas from [8].

We index the closed orbits such that $\lambda(\tau_1) \leqq \lambda(\tau_2) \leqq \cdots$. Choose integers l, k (with $k > 0$) and $\delta > 0$ and define

$$F^l_{k,\delta} = F_\delta = \{\alpha \,\|\, |f_k(\alpha)| > \delta_k\} \quad \text{where } f_k(\alpha) = f_k(\alpha, l) = \sum_{j=1}^{k} e(l\alpha \exp h\lambda(\tau_j)).$$

We want to estimate the size of F_δ. If $\Phi^\delta(t) \leqq C_0 t^\theta$, for $t > 0$ (for sufficiently large C_0), then for $n > 0$ we have $\Phi^{n\delta}(t) \leqq nC_0 t^\theta$. We can now estimate

$$\begin{aligned}
\int_0^1 |f_k(\alpha)|^2 d\alpha &= \sum_{j,i=1}^{k} \int_0^1 e(l\alpha[\exp h\lambda(\tau_j) - \exp h\lambda(\tau_i)])d\alpha \\
&\leqq \sum_{j,i=1}^{k} \min\{1, 2/[2\pi l\,|\exp h\lambda(\tau_j) - \exp h\lambda(\tau_i)|]\} \\
&\ll (\log k)k^{\theta+1} \qquad (4.1)\\
&\ll k^{\theta_0+1} \qquad \text{for any } \theta < \theta_0 < 1.
\end{aligned}$$

In particular, the Lebesgue measure of F_δ is of order $k^{-(1-\theta_0)}$.

We next want to construct a cover F_δ subject to certain constraints. Choose $\eta > 1$; then since, by Proposition 4, $\pi(t) \sim t/\log t$ we have $C = C(\eta) > 0$ such that $\exp h\lambda(\tau_r) \leqq Cr^\eta$ [10]. This estimate allows us to write

$$|df_k/d\alpha| \leqq 2\pi lC \sum_{j=1}^{k} j^\eta \leqq 2\pi lC \cdot k^{\eta+1}. \tag{4.2}$$

For any $x_0 \in F_\delta$, (4.2) gives that we can choose intervals about x_0, contained wholly in $F_{\delta/2} \supseteq F_\delta$, of length greater than

$$2 \cdot (\delta k/2)/(2\pi lC)k^{\eta+1} \gg k^{-\eta}.$$

Furthermore, the above condition (4.1) on $F_{\delta/2}$ restricts the size of such intervals to have maximum length of order $k^{-(1-\theta_0)}$.

Thus, we can cover F_δ by $N_\delta = N_\delta(l, k)$ such intervals of lengths $l_1, \ldots, l_N$ where

$$N_\delta \cdot k^{-\eta} \ll k^{-(1-\theta_0)}, \qquad \text{i.e. } N_\delta \ll k^{\eta-(1-\theta_0)}.$$

It follows from convexity of $t \to t^\rho$, for $0 \leqq \rho \leqq 1$, that

$$\sum_{i=1}^{N_\delta} (l_i)^\rho \ll N_\delta \left(\frac{k^{-(1-\theta_0)}}{N_\delta} \right)^\rho \ll k^\varepsilon$$

where $\varepsilon = \eta(1-\rho) - (1-\theta_0)$. In particular, this exponent is negative if $\rho < 1-(1-\theta_0)/\eta$.

Introduce

$$S = \bigcup_{l=-\infty}^{+\infty} \bigcup_{r=1}^{+\infty} \bigcap_{q \geqq |l|+r} \bigcup_{m=q}^{+\infty} F^l_{a_m, 1/r},$$

where $a_m = [\exp(m/\log m)]$. Using the above construction we have a cover $\mathscr{C}$ (of arbitrarily small size) for which

$$\sum_{C \in \mathscr{C}} (\operatorname{diam} C)^\rho \ll \sum_{\substack{l,r \\ m \geqq |l|+r}} l^{1-\rho} \cdot r^{5-\rho} \cdot \exp -(\varepsilon m/\log m) < +\infty.$$

Since $E \subseteq S$ we conclude that $\mathrm{HD}(E) \leqq 1-(1-\theta_0)/\eta$. Because $\theta_0 > \theta$ and $\eta > 1$ can be chosen arbitrarily, the result follows.

COROLLARY 5.1. (a) *If there exists $k > 2$ for which $\pi(t) = \mathrm{li}(t) + O(t/(\log t)^k)$ then E has zero Lebesgue measure.*

(b) *If there exists $0 < \theta < 1$ for which $\pi(t) = \mathrm{li}(t) + O(t^\theta)$ then $\mathrm{HD}(E) \leqq \theta$.*

It is illuminating to consider the following simple example:

EXAMPLE. Let $\Sigma = \Pi_{-\infty}^{+\infty}\{0,1\}$ and define the function $f: \Sigma \to R$ by

$$f(x) = \begin{cases} \log 2 & \text{if } x_0 = 0, \\ \log(3/2) & \text{if } x_0 = 1. \end{cases}$$

For the suspended flow σ^f, which has entropy $h(\sigma^f) = 1$, the sequence $(\exp \lambda(\tau))$ takes values $3^{n+m}/2^n$, $n, m > 0$, with multiplicity $\binom{n+m}{n}$. Furthermore, there exists no choice of $0 < \theta < 1$ for which $\pi(t) = t/\log t + O(t^\theta)$. However, by choosing $\delta > 0$ sufficiently small and appealing to Stirling's asymptotic formula for factorials we get

$$\Phi^\delta(t) \ll t^\varepsilon/(\log t)^{1/2} \qquad \text{where } \varepsilon = \log 4/\log(9/2).$$

Finally, we remark that it is possible to find uncountably many $0 < \alpha < 1$ for which $(\alpha \exp \lambda(\tau) \pmod 1)$ is *not* uniformly distributed (such α may be constructed through their triadic expansions (cf. [9])).

We cannot ask similar questions regarding uniform distribution with $\lambda(\tau)$ replacing $\exp h\lambda(\tau)$ since the asymptotic rate of growth means that no sequence $(\rho\lambda(\tau) \pmod 1)$ is uniformly distributed for any $\rho > 0$. However, for geodesic flows on compact surfaces of constant negative curvature we have associated to a geodesic γ not only its length $l(\gamma)$ but also its word length $\omega(\gamma)$.

PROPOSITION 6. *Let W_n be the set of all closed geodesics of word length $\omega(\gamma) = n$ and let $\rho > 0$. Define μ_n to be the (purely atomic) probability measure on the unit interval formed by equidistributing measure over the fractional lengths $(\rho l(\gamma))$ of $\rho l(\gamma)$, for $\gamma \in W_n$, i.e.*

$$\mu_n = \frac{1}{\operatorname{Card} W_n} \sum_{\gamma \in W^n} \delta_{(\rho l(\gamma))}.$$

Then μ_n converges to Lebesgue measure in the weak topology.*

PROOF. Using symbolic dynamics this proposition reduces to a result on suspended flows [20], [16]. From the remarks in section 3 we have that W_n corresponds to closed σ-orbits of period n and for $\gamma \in W_n$ corresponding to $\{x, \sigma x, \dots, \sigma^{n-1}x\}$ we have $l(\gamma) = f^n(x)$. Furthermore, for $k \in Z\setminus\{0\}$,

$$\int_0^1 e(kx)d\mu_n(x) \ll \sum_{\sigma^n x = x} e(kf^n(x))/\operatorname{Card}\{x \mid \sigma^n x = x\}. \tag{4.3}$$

However, from [17] it follows that the right-hand side of (4.3) tends to zero as n increases.

REMARK. By analogy with a result of Siepinski [21] for primes we have the following corollary to Proposition 6: Given any finite sequence $a_0, \dots, a_{k-1} \in \{0, 1, \dots, 9\}$, $a_0 \neq 0$, there exist infinitely many geodesics for which the first k-terms in the decimal expansion of $[\exp \alpha l(\gamma)]$ are $a_1 \cdots a_k$ (for any $\alpha > 0$).

REMARK. For the modular surface the values $\exp l(\gamma)$ are given by $m^2 + (m^2 - 1/4)^{1/2} = 2m^2 + 2 + O(1/m^2)$, $m \geqq 1$ [11], and defining E as before, $\mathrm{HD}(E) \leqq 1/2$.

REMARK. Another device for studying lengths of closed orbits is to consider the limit points S of the sequence $\log(x_{n+1} - x_n)/\log x_n$ where $x_n = \exp h\lambda(\tau)$. This gives an indication of the size of gaps which can occur between successive x_n. (If we take x_n to be the nth prime number then the limit points are bounded above by 7/12.) For the modular surface we have more exact knowledge of orbit lengths and $S = \{-\infty, 1/2\}$ (where $-\infty$ arises from distinct orbits with the same lengths). If we consider a locally constant suspended flow whose height function takes two values α and β with α/β algebraic, then Roth's theorem for diophantine approximation of algebraic numbers allows us to obtain a lower bound on $0 < \exp(k\alpha + l\beta)h$ for $k, l = O(N)$ and thus prove that $\{-\infty, 1\} \subseteq S$.

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